Proper initial conditions for the lubrication model of the flow of a thin film of fluid

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Abstract

A lubrication model describes the dynamics of a thin layer of fluid spreading over a solid substrate. But to make forecasts we need to supply correct initial conditions to the model. Remarkably, the initial fluid thickness is *not* the correct initial thickness for the lubrication model. Theory recently developed in [12, 14] provides the correct projection of initial conditions onto a model of a dynamical system. The correct projection is determined by requiring that the model's solution exponentially quickly approaches that of the actual fluid dynamics. For lubrication we show that although the initial free surface shape contributes the most to the model's initial conditions, the initial velocity field is also an influence. The projection also gives a rationale for incorporating miscellaneous small forcing effects into the lubrication model; gravitational forcing is given as one example.

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Contents

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1   Introduction

The flows of thin films of fluids are encountered in many engineering and
biological applications. They include: the flow of rainwater on a road or
windscreen or other draining problems [2]; paint and coating flows [17, 18];
the flow of many protective biological fluids [6]; and other coating, painting
and drying processes [10, 8, 20, e.g.]. The fluid film thickness and the average
fluid flux are the main characteristics of interest in these applications. The
fine details of the actual local velocity and pressure fields typically are of less
practical importance. For this reason the various approximations have been
constructed over the past decades [2, 11, 18, 13, 7, e.g.] to model the evo-
lution of the fluid flow in various geometries [16] and parameter regimes [5].
We consider herein the basic nondimensional lubrication model for surface
tension dominated flows,

$$\eta_t \approx -\frac{1}{3}\partial_x\left(\eta^3\eta_{xxx}\right) , \tag{1}$$

where $\eta(x,t)$ is the thickness of the fluid film spreading over a solid substrate
(at $y=0$). Centre manifold theory [1] provides a generic and systematic pro-
cedure for deriving such models for a wide variety of fluid flows. Recently,
Roberts *et al.* [15, 16] showed how this well established lubrication model of
thin film flow is rigorously derived from the governing Navier-Stokes equa-
tions (outlined in Section 2) using a computer algebra implementation of
centre manifold techniques. The model is derived under the assumption that
longitudinal derivatives, $\partial/\partial x$, are small—the slowly varying assumption—
as used extensively in creating models of shear dispersion, [9, 19, e.g.]. The
centre manifold based algorithm provides a straightforward derivation of the
model up to arbitrarily high order [15], but in this work we limit ourselves
to consideration of the above leading order model.

   It is frequently believed that the initial conditions for the models of long-
term evolution are not highly important because the asymptotic state does
not depend on a character of transient processes in the system. It was shown
in [12, 4] that initial conditions *can have* a long-lasting influence on forecasts.
This seems especially true for models of spatio-temporal dynamics such as

the lubrication model. Similarly, in the case of dispersion in a channel [12] or pipe [9] the long term location and spread of a pollutant does depend upon the details of the initial release of the pollutant. Thus the initial conditions for a model must be chosen carefully to ensure the long term fidelity between the model and the physical flow. It will be shown in Section 3 that the initial form of the free surface for the lubrication model should be different from the initial thickness of the physical fluid. In particular, if the fluid initially has thickness $\eta_0(x)$ but zero velocity and pressure then the lubrication model (1) should be solved with initial condition that $\eta(x,0) = h_0(x)$ where

$$h_0 \approx \eta_0 + \eta_0 \eta_{0xx} \,. \tag{2}$$

In general, the initial condition h_0 for the model is the non-trivial function of initial velocities and pressure distributions given by (34)–(35). The argument for these initial conditions is based upon the dynamics near the low-dimensional centre manifold, that is, upon the physics of the approach to the lubrication model. The general arguments, developed in [12] and recently refined in [14], are based upon the geometric picture provided by centre manifold theory. The principle aim of this paper is to apply this general framework to the considerable complications of the infinite dimensional dynamics of thin film fluid flow and so derive (2) and its generalisations. This is the first time that correct initial conditions have been obtained for lubrication theory.

An interesting aspect of the general analysis developed in [12] and [14] is that the projection of initial conditions also gives a rationale for treating small forcing of the dynamics. This connection was more fully explored by Cox & Roberts [3] who discussed the effects upon the centre manifold and the evolution thereon for time dependent forcing. In Section 4 we apply the projection to a gravitational forcing of the thin fluid layer to verify the veracity of the classic model

$$\eta_t \approx -\frac{1}{3}\partial_x \left[\eta^3 \left(\eta_{xxx} + B \sin\theta - B \cos\theta \, \eta_x \right) \right] , \tag{3}$$

where B is the nondimensional magnitude of gravity and θ is the downwards angle of the substrate. This is derived here as just one example of a very general result that applies to all small forcing effects upon the fluid flow.

2 The lubrication model of fluid flow

We consider a two-dimensional flow of thin film of Newtonian fluid along a flat horizontal substrate. The free surface is given by $y = \eta(x,t)$, where x and

y are horizontal and vertical coordinates respectively. The flow, with velocity $\mathbf{q} = (u, v)$ and pressure p, is governed by the incompressible Navier-Stokes equations

$$\mathbf{q}_t + \mathbf{q} \cdot \nabla \mathbf{q} \;=\; -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{q}, \tag{4}$$

supplemented by the continuity equation

$$\nabla \cdot \mathbf{q} \;=\; 0, \tag{5}$$

non-slip boundary conditions on the bottom

$$\mathbf{q} = \mathbf{0} \quad \text{on } y = 0, \tag{6}$$

and tangential stress and normal stress conditions on the free surface

$$2\eta_x(v_y - u_x) + \left(1 - \eta_x^2\right)(u_y + v_x) = 0 \quad \text{on } y = \eta, \tag{7}$$

$$\left(1 + \eta_x^2\right) p = 2\mu \left[v_y + \eta_x^2 u_x - \eta_x (u_y + v_x)\right]$$
$$- \frac{\sigma \eta_{xx}}{\sqrt{1 + \eta_x^2}} \quad \text{on } y = \eta, \tag{8}$$

respectively, as discussed in detail by Roberts [15]. We close the problem with the kinematic condition relating the velocity of the fluid on the surface to the evolution of the free surface:

$$\eta_t = v - u\eta_x \quad \text{on } y = \eta. \tag{9}$$

In the above equations ρ is the fluid density, ν is the kinematic viscosity, and σ is the coefficient of the surface tension. The fluid film is assumed to be so thin that the gravity force in the momentum equations can be neglected (see the discussion in [15]) at least initially.

We non-dimensionalise the governing equations using a typical film thickness H as a reference length, reference time $\mu H/\sigma$ (where μ is the dynamic viscosity of the fluid), reference speed σ/μ, and reference pressure σ/H. On this small scale, fluid viscosity is strong and the fluid layer is of very large extent laterally. The non-dimensional Navier-Stokes and continuity equations then become

$$\mathcal{R}\left(\mathbf{q}_t + \mathbf{q} \cdot \nabla \mathbf{q}\right) \;=\; -\nabla p + \nabla^2 \mathbf{q}, \tag{10}$$
$$\nabla \cdot \mathbf{q} \;=\; 0, \tag{11}$$

where $\mathcal{R} = \rho\sigma H/\mu^2$ is a Reynolds number. They are complemented by the normal stress condition

$$\left(1 + \eta_x^2\right) p = 2\left[v_y + \eta_x^2 u_x - \eta_x\left(u_y + v_x\right)\right] - \frac{\eta_{xx}}{\sqrt{1 + \eta_x^2}} \quad \text{on } y = \eta , \qquad (12)$$

and equations (6), (7) and (9) which remain symbolically unchanged under the nondimensionalisation.

In such a thin layer of fluid, the infinite number of horizontal shear modes decay exponentially quickly through viscous dissipation acting across the thin film. Thus in the long term, the dynamics are driven by surface tension trying to flatten surface curvature. Centre manifold theory [1] is used in such circumstances to systematically derive the low-dimensional model of the long term evolution, here the lubrication model (1) for the fluid layer's thickness η, see [15, §3] for more introductory detail. The approximate form of the lubrication model for such a flow is obtained as a formal expansion in orders of the x-derivatives under the assumption that these derivatives are small. Although the model can be developed to an arbitrary order of spatial derivatives using the iterative computer algebra algorithm suggested in [15], the expressions for higher order approximations are very involved and thus we present here only the lowest order model. To errors of fifth-order in ∂_x and parameterized by the free surface thickness η, the centre manifold $\mathbf{v} = (u(\eta), v(\eta), p(\eta), \eta)$ is given by

$$u \approx \left(y\eta - \frac{1}{2}y^2\right)\eta_{xxx} , \qquad (13)$$

$$v \approx -\frac{1}{2}y^2\eta_x\eta_{xxx} + \left(\frac{1}{6}y^3 - \frac{1}{2}y^2\eta\right)\partial_x^4\eta , \qquad (14)$$

$$p \approx -\eta_{xx} + \frac{3}{2}\eta_x^2\eta_{xx} - (\eta + y)\eta_x\eta_{xxx} - \left(\frac{1}{2}\eta^2 + y\eta - \frac{1}{2}y^2\right)\partial_x^4\eta , \quad (15)$$

where η evolves according to (1). Observe that up to this order the model does not depend on the Reynolds number—fluid inertia is negligible. The lubrication model (1) is the basic model for the dynamics of thin fluid films.

3 Project the initial conditions

In order to use the lubrication model (1) to make forecasts, it should be supplemented with initial conditions. Roberts [12] has shown that determining the correct initial conditions is a nontrivial problem. Remarkably, in general the initial value of η for model (1) differs from the initial fluid thickness for the physical problem (6)–(12). To distinguish between the two, we denote

the initial fluid thickness by η_0 and use h_0 to denote the initial conditions for model (1) of the fluid's evolution. The main task of this paper is to show how to determine h_0 as a function of the initial fluid state.

To define the proper initial conditions for model (1) we follow the procedure outlined by Roberts [14] and examine the dynamics in the vicinity of the centre manifold. We start by linearizing the governing equations about the centre manifold (13)–(15) by writing the fluid variables as the sum $(u, v, p, \eta) + (u', v', p', \eta')$ where primed quantities are the assumed small displacement from the centre manifold, and so their products are neglected. The resulting system is

$$\mathcal{R}\left(\mathbf{q}'_t + \mathbf{q}' \cdot \nabla \mathbf{q} + \mathbf{q} \cdot \nabla \mathbf{q}'\right) + \nabla p' - \nabla^2 \mathbf{q}' = 0 \,, \tag{16}$$

$$\nabla \cdot \mathbf{q}' = 0 \,, \tag{17}$$

with the boundary conditions at $y = \eta$

$$\eta'_t - v' + u' \eta_x + u \eta'_x + u_y \eta_x \eta' - v_y \eta' = 0 \,, \tag{18}$$

$$2\eta'_x (v_y - u_x) + 2\eta_x \left(v'_y - u'_x\right) + \left(1 - \eta_x^2\right)\left(u'_y + v'_x\right) - 2\eta_x \eta'_x (u_y + v_x)$$
$$+ 2\eta_x (v_{yy} - u_{xy}) \eta' + \left(1 - \eta_x^2\right)(u_{yy} + v_{xy}) \eta' = 0 \,, \tag{19}$$

$$\left(1 + \eta_x^2\right) p' + 2\eta_x \eta'_x p + \left(1 + \eta_x^2\right) p_y \eta' + \frac{\eta'_{xx}}{\sqrt{1 + \eta_x^2}} - \frac{\eta_{xx} \eta_x \eta'_x}{(1 + \eta_x^2)^{3/2}} =$$
$$2\left[v'_y + \eta_x^2 u'_x + 2\eta_x \eta'_x u_x - \eta_x \left(u'_y + v'_x\right) - \eta'_x (u_y + v_x)\right] \tag{20}$$
$$+ 2\left[v_{yy} + \eta_x^2 u_{xy} - \eta_x (u_{yy} + v_{xy})\right] \eta' \,,$$

and the homogeneous boundary conditions for the velocity $\mathbf{q}' = \mathbf{0}$ at $y = 0$. The above equations describe the dynamics of the fluid near the centre manifold (13)–(15).

Typically, the initial conditions $\mathbf{u}_0 = (u_0(x, y), v_0(x, y), p_0(x, y), \eta_0(x))$ for the original fluid layer equations (6)–(12) do not belong to the low-dimensional centre manifold $\mathbf{v}$ given by (13)–(15). Thus they cannot be used directly as a starting point for model (1). As shown in [12] and [14] the proper model initial condition is the projection $\mathbf{v}_0 = (u(h_0), v(h_0), p(h_0), h_0)$ from $\mathbf{u}_0$ to the centre manifold along the isochron—in the state space an isochron is a surface of all the initial states which have the same long-term dynamics on the centre manifold (up to an exponentially small error). Consequently, the model initial conditions are determined to satisfy

$$\langle \mathbf{z}, \mathbf{u}_0 - \mathbf{v}_0 \rangle = 0 \,, \tag{21}$$

where $\mathbf{z} = (u^\dagger, v^\dagger, p^\dagger, \eta^\dagger)$ is a vector orthogonal to the direction of projection (the dagger is used to denote field quantities in the adjoint space). Here the inner product is defined for four component vector fields

$$\mathbf{a} = (a^1(x,y,t), a^2(x,y,t), a^3(x,y,t), a^4(x,t)) \quad \text{and}$$
$$\mathbf{b} = (b^1(x,y,t), b^2(x,y,t), b^3(x,y,t), b^4(x,t))$$

as

$$\langle \mathbf{a}, \mathbf{b} \rangle \equiv \int_{-\infty}^{\infty} \int_0^\eta (a^1 b^1 + a^2 b^2 + a^3 b^3)\, dy\, dx + \int_{-\infty}^{\infty} a^4 b^4\, dx\,. \tag{22}$$

According to the arguments developed in [12] and refined in [14], the defining vector of the projection, $\mathbf{z}$, satisfies the dual equation

$$\mathcal{D}\mathbf{z} = \langle \mathcal{D}\mathbf{z}, \mathbf{e} \rangle \mathbf{z}\,, \tag{23}$$

where $\mathbf{e}$ is the local tangent vector to the centre manifold

$$\mathbf{e} = \frac{\partial}{\partial \eta} \begin{bmatrix} u \\ v \\ p \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\partial_x^2 \\ 1 \end{bmatrix} + \mathcal{O}\left(\partial_x^3\right)\,, \tag{24}$$

and the dual operator $\mathcal{D}$ is obtained from equations adjoint to (16)–(20) with respect to the inner product defined as

$$\langle\langle \mathbf{a}, \mathbf{b} \rangle\rangle \equiv \int_0^t \langle \mathbf{a}, \mathbf{b} \rangle\, d\tau\,. \tag{25}$$

Higher order derivative terms in (24) can be easily computed but, as will be shown later, the given second order truncation will suffice for finding the initial conditions to the first few orders. Note that the local tangent to the centre manifold is a vector operator rather than just a vector function as occurs in the finite dimensional cases discussed in [14]. Being introduced into the inner product (22), it acts on the other vector involved before the integration is performed. Using the above inner products, the adjoint expressions of (16)–(20) leading to the dual operator $\mathcal{D}$ are:

$$\mathcal{D}\mathbf{z} = \begin{bmatrix} \mathcal{R}\left(u_t^\dagger - u^\dagger u_x - v^\dagger v_x + u u_x^\dagger + v u_y^\dagger\right) + p_x^\dagger + u_{xx}^\dagger + u_{yy}^\dagger \\ \mathcal{R}\left(v_t^\dagger - u^\dagger u_y - v^\dagger v_y + u v_x^\dagger + v v_y^\dagger\right) + p_y^\dagger + v_{xx}^\dagger + v_{yy}^\dagger \\ u_x^\dagger + v_y^\dagger \\ \eta_t^\dagger + \eta^\dagger - p^\dagger + v^\dagger \eta_{xx} + u_x^\dagger + 2v_x^\dagger \eta_x - v_y^\dagger \quad \text{on } y = \eta \end{bmatrix}\,. \tag{26}$$

The adjoint velocities satisfy homogeneous boundary conditions $\mathbf{q}^\dagger = \mathbf{0}$ at $y = 0$. The adjoint boundary conditions for the velocities at $y = \eta$ are

represented by expressions too long to be given here. Instead in the Appendix we include computer algebra code written in REDUCE for obtaining them and the dual operator $\mathcal{D}$. Periodic boundary conditions at $x = \pm\infty$ are used in the derivation.

Given the above dual $\mathcal{D}$, system (23) is solved asymptotically assuming that it is possible to neglect higher order derivatives with respect to x. The treatment of ∂_x as small is equivalent to the assumption of slow variation in x. The iterative algorithm is quite similar to the one described in [15] and used to derive the centre manifold model (13)–(15) & (1). Thus here we just make a few notes on the specifics of the first few iterations. In essence the procedure is as follows: we start by solving the equations neglecting all x derivatives and then in further iterations compute the corrections associated with these derivatives of functions found at previous iterations. Owing to the special form of the vector $\mathbf{e}$, the right-hand side of (23) remains zero during the first few iterations required to obtain the leading order (in ∂_x) expressions for $\mathbf{z}$. It is easier to first look for the solution $\mathbf{z}$ in the functional

$$
\mathbf{z} =
\begin{bmatrix}
u^\dagger \\
v^\dagger \\
p^\dagger \\
\eta^\dagger
\end{bmatrix}
\approx
\begin{bmatrix}
0 \\
0 \\
\eta^\dagger \\
\eta^\dagger
\end{bmatrix}
+
\begin{bmatrix}
\eta^\dagger_x \left(y\eta - \frac{1}{2}y^2 \right) \\
0 \\
0 \\
0
\end{bmatrix}
\tag{27}
$$

$$
+
\begin{bmatrix}
0 \\
\eta^\dagger_{xx} \left(\frac{1}{6}y^3 - \frac{1}{2}y^2\eta \right) - \frac{1}{2}\eta^\dagger_x \eta_x y^2 \\
\eta^\dagger_{xx} \left(y\eta - \frac{1}{2}y^2 \right) + \eta^\dagger_x \eta_x y + \frac{1}{2}\partial_x \left(\eta^\dagger_x \eta^2 \right) \\
0
\end{bmatrix},
$$

$$
\text{s.t.} \quad \eta^\dagger_t \approx 0.
\tag{28}
$$

From the structure of the successive corrections to the solution for $\mathbf{z}$ we deduce that the initial conditions for the model are influenced the most by the initial form of the fluid surface. This is the expected result for such a surface tension dominated flow. The initial horizontal velocity field has a secondary effect on the flow (corresponding terms in (27) appear only in the second iteration) primarily as a response to the horizontal pressure gradient induced by the surface curvature. The vertical motion is even less important since it is severely restricted by the small thickness of the fluid layer.

An additional condition must be exploited to determine $\eta^\dagger$ as an asymptotic expansion in ∂_x. It is done using the normalisation condition $\langle \mathbf{z}, \mathbf{e} \rangle = 1$ (see [14]) which upon (24) leads to

$$
\int_{-\infty}^{\infty} \left(\eta^\dagger - \int_0^\eta p^\dagger_{xx}\, dy \right) dx + \mathcal{O}\left(\partial_x^3 \right) = 1.
\tag{29}
$$

At the leading order we then obtain

$$\int_{-\infty}^{\infty} \eta^{\dagger} \, dx = 1 \, . \tag{30}$$

This condition does not provide any unique solution for $\eta^{\dagger}$ but rather a continuum of linearly independent localised functions. Without any loss of generality we choose the linearly independent solutions

$$\eta^{\dagger}(x; x_*) = \delta(x - x_*) + \mathcal{O}\left(\partial_x^2\right) \, . \tag{31}$$

where x_* is an arbitrary point and δ is the Dirac delta function. At the next iteration we require the second order contribution to (29) to vanish. This results in

$$\eta^{\dagger}(x; x_*) = \delta(x - x_*) + \delta''(x - x_*)\eta(x, t) + o\left(\partial_x^2\right) \, . \tag{32}$$

This expression for $\eta^{\dagger}$ is used in (27) to determine $\mathbf{z}$, the defining vector of the proper projection onto the centre manifold.

Lastly, we use $\mathbf{z}$ in (21) to project an initial condition. Requiring the integrand in equation (21) to vanish and taking into account (27), (31) and (32) we obtain

$$\eta_0 - h_0 + \overline{p_0} + \partial_x \left[\overline{u_0 y \left(\tfrac{y}{2} - h_0\right)}\right]$$
$$-\partial_x \left[h_{0x} \left(\overline{p_0(h_0 + y) - \tfrac{1}{2}v_0 y^2}\right)\right] + h_0 h_{0xx} \tag{33}$$
$$+\partial_x^2 \left[\overline{h_0 \eta_0 - h_0^2 + p_0 \left(\tfrac{h_0^2 - y^2}{2} + h_0(y + 1)\right) - \tfrac{v_0}{2}\left(h_0 y^2 - \tfrac{y^3}{3}\right)}\right] \approx 0 \, ,$$

where the notation $\overline{f} \equiv \int_0^{\eta_0} f \, dy$ is introduced. This equation determines h_0 and can be solved iteratively as well. The first three iterations produce

$$h_0 \approx h_{00} + h_{01} + h_{02} \, , \tag{34}$$

where

$$
\begin{aligned}
h_{00} &= \eta_0 + \overline{p_0} \, , \\
h_{01} &= \partial_x \left[\overline{u_0 y \left(\frac{y}{2} - h_{00}\right)}\right] \, , \\
h_{02} &= -\partial_x [h_{01}\overline{u_0 y}] + h_{00}h_{00xx} - \partial_x \left[h_{00x}\left(\overline{p_0(h_{00} + y) - \frac{1}{2}v_0 y^2}\right)\right] \\
&\quad + \partial_x^2 \left[\overline{p_0 \left(\frac{h_{00}^2 - y^2}{2} + h_{00}y\right) - \frac{v_0}{2}\left(h_{00}y^2 - \frac{y^3}{3}\right)}\right] \, .
\end{aligned}
\tag{35}
$$

Note some specific cases of interest.

- Parallel shear flow $v_0 = p_0 = 0$, $\eta_0 = 1$ and $u_0 = u_0(y)$. Accord-
  ing to the linear stability analysis, due to viscous dissipation such
  a flow approaches the motionless state exponentially quickly. Thus
  the centre manifold model, which disregards exponentially fast tran-
  sients, must give rise to a stationary solution. In this case the initial
  conditions (34) for the model and the model solution itself are just
  $\mathbf{v}_0 = \mathbf{v}(t) = (0, 0, 0, 1)$ and do indeed correspond to the motionless
  uniform fluid film.

- Initially stationary fluid layer ($u_0 = v_0 = 0$) with uniform pressure
  equal to the atmospheric one ($p_0 = 0$) and curved free surface $\eta_0 \neq$
  const. In this case $h_0 \approx \eta_0(1 + \eta_{0xx})$ and the initial conditions for
  the model coincide with the initial film thickness only to leading or-
  der. Higher order terms tend to smooth out the initial distribution of
  the model film thickness flattening "hills" and "valleys". This can be
  interpreted in the following way. The physical fluid which is initially
  motionless requires time for acceleration i.e. time to approach the centre
  manifold (13)–(15) in which velocities are generally non-zero. Dissipa-
  tion during this transient acceleration leads to a decrease in the energy
  of the system. Since the energy of the system up to leading order in
  ∂_x is just the potential energy associated with the surface tension and
  is proportional to the surface curvature the initial condition models
  the energy loss by levelling out the free surface in comparison with the
  original distribution.

- Nonzero initial average pressure. It leads to a change in the initial
  model fluid film thickness when compared with that of the original
  problem. In particular, locally positive initial pressure corresponds to
  locally thicker fluid film in the model. This is intuitively expected
  since the increased pressure inside the fluid layer (imagine an under-
  water explosion) acts against the local surface tension and leads to the
  appearance of the local "hill" on the film surface.

Finally we note that the initial condition for the model is most sensitive
to the fluid film thickness and the local pressure whereas the initial velocity
field has just secondary effect on the long term film dynamics. This is not a
surprise since, as noted in [15], the considered flow is essentially the creeping
one and inertia effects are less important than the influence of the surface
tension or, equivalently, of the surface curvature.

4   Gravitational forcing as an afterthought

In the previous sections the influence of gravity on the thin film flow is neglected. Here we demonstrate how such a forcing may be added into the model using the projection derived for initial conditions as argued in general in [14]. The general technique may be used to include physical processes into the lubrication model *after* developing the model. The result given here for gravity is just one specific example.

The correction to the model accounting for the gravity can be obtained by iterative solution of the Navier-Stokes equations (10) where terms $\mathbf{f} = B(g_1, g_2)$ responsible for gravity are introduced in the right-hand side [16, e.g.]. Here $B = \rho|\mathbf{g}|H^2/\sigma$ (assumed finite but small) is the Bond number [15] and g_1 and g_2 are components of the non-dimensional gravity vector in x- and y-directions, respectively. Alternatively, considering gravity as a specific example of a forcing which by arguments in [14] can be directly projected onto the model (1). Geometrically, the centre manifold obtained for the dynamics without forcing is deformed slightly when forcing is applied such that each point of the unforced centre manifold is shifted along the isochron passing through the original location of this point as discussed by Cox & Roberts [3]. According to [14] the dynamics of the free surface subject to forcing is described by the modified model (1)

$$\eta_t \;\approx\; -\frac{1}{3}\partial_x\left(\eta^3\eta_{xxx}\right) + q\,, \tag{36}$$

where q is the projection of the forcing $\mathbf{f}$ of the fluid, namely

$$q = \langle \mathbf{z}, \mathbf{f}\rangle\,. \tag{37}$$

Typically, gravity is uniform in thin film flow applications and then $(g_1, g_2) = (\sin\theta, -\cos\theta)$, where θ is the downwards angle between a flat substrate and the horizontal. Then upon using (27) and (31), (37) immediately leads to

$$\begin{aligned}
q \;=\; & B\int_{-\infty}^{\infty}\int_0^{\eta}\left\{g_1\delta_x'\left(\eta - \frac{1}{2}y\right)y\right. \\
& \left. + \frac{1}{2}g_2\left[\delta_{xx}''\left(\frac{1}{3}y - \eta\right) - \delta_x'\eta_x\right]y^2\right\}dy\,dx \\
\;=\; & -B\left[g_1\eta^2\eta_x + \frac{1}{3}g_2\partial_x\left(\eta^3\eta_x\right)\right]\,,
\end{aligned} \tag{38}$$

which is identical to the correction obtained from directly modelling the forced equations (10) through assuming small Bond number [16]. The theory of initial conditions recently refined in [14] also provides an elegant way of modifying the model to incorporate forcing.

5   Conclusions

The proper initial conditions for the lubrication model of flow of thin film is derived using the projection of the initial conditions for the original problem onto the centre manifold representing the lubrication model. The obtained results are easily generalised to the case of isotropic three-dimensional thin film flow. Then the two-dimensional lubrication model

$$\eta_t \approx -\frac{1}{3}\nabla \cdot \left(\eta^3 \nabla^3 \eta\right) \tag{39}$$

should be solved with initial conditions given up to the first order by

$$h_0 \approx \eta_0 + \overline{p_0} + \nabla \cdot \overline{\mathbf{q}_0 y \left(\frac{1}{2} y - \eta_0 - \overline{p_0}\right)}. \tag{40}$$

Here $\mathbf{q}_0 = (u_0, w_0)$ is the initial horizontal velocity field for the original problem and ∇ is a two-dimensional operator in xz-plane.

A   Computer algebra derives the dual

REDUCE[1] code for determining the dual operator $\mathcal{D}$ along with its associated boundary conditions:

```
1   % Seek to find adjoint of thin fluid film equations.
2   % Linearise about known centre manifold vu,vv,vp,vh.
3   % a denotes adjoint quantities, b - physical ones.
4   factor b; on div; off allfac; % improves the output text
5   operator b,a;
6   depend b,x,y,t$ depend a,x,y,t$ depend vh,x,t$
7   depend vu,x,y,t$ depend vv,x,y,t$ depend vp,x,y,t$
8   % look at linearisation about centre manifold dynamics
9   let del^2=>0$ % gets rid of all nonlinear terms
10  qu:=vu+del*b(u)$ qv:=vv+del*b(v)$
11  pp:=vp+del*b(p)$ hh:=vh+del*b(h)$
12  % some shorthands
13  depend rsqrt,x,t$ % short for 1/sqrt(1+h_x^2)
```

---

[1]At the time of writing, information about reduce was available from Anthony C. Hearn, RAND, Santa Monica, CA 90407-2138, USA. E-mail: reduce@rand.org

```
14  depend rmin,x,t$  % short for 1/(1-h_x^2)
15  let { df(vh,x)^2*rsqrt^2 => 1-rsqrt^2
16      , df(vh,x)^2*rmin    => rmin-1
17      , (df(vh,x)^2+1) => 1/rsqrt^2
18      , df(rsqrt,~zz) => -rsqrt^3*df(vh,x)*df(vh,x,zz)
19      , df(rmin,~zz)  => 2*rmin^2*df(vh,x)*df(vh,x,zz)
20      , rmin*rsqrt^2  => (rmin+rsqrt^2)/2 }$
21  % physical equations
22  umom:=re*(df(qu,t)+qu*df(qu,x)+qv*df(qu,y))+df(pp,x)
23       -df(qu,x,x)-df(qu,y,y)$
24  vmom:=re*(df(qv,t)+qu*df(qv,x)+qv*df(qv,y))+df(pp,y)
25       -df(qv,x,x)-df(qv,y,y)$
26  cty:=df(qu,x)+df(qv,y)$
27  fkin:=df(hh,t)-qv+qu*df(hh,x)$
28  ftan:=2*df(hh,x)*(df(qv,y)-df(qu,x))
29       +(1-df(hh,x)^2)*(df(qu,y)+df(qv,x))$
30  fnor:=(1+df(hh,x)^2)*pp
31       -2*(df(qv,y)+df(hh,x)^2*df(qu,x)
32       -df(hh,x)*(df(qu,y)+df(qv,x)))
33       +df(hh,x,x)/sqrt(1+df(hh,x)^2)$
34  % linearization of boundary and kinematic conditions
35  let{df(vv,y)=>-df(vu,x),df(b(h),y) =>0}$
36  fkin:=df(fkin+del*b(h)*df(fkin,y),del)$
37  rl1:={df(b(v),y)=>-df(b(u),x)}$
38  let rl1$
39  ftan:=rmin*df(ftan+del*b(h)*df(ftan,y),del)$
40  fnor:=rsqrt^2*sub(del=0,df(fnor+del*b(h)*df(fnor,y),del));
41  clearrules rl1$
42  let {df(vh,t)=>vv-vu*df(vh,x)}$
43  % linearized equations
44  umom:=df(umom,del)$ vmom:=df(vmom,del)$ cty:=df(cty,del)$
45  % innerproduct form and integrate
46  on list$
47  operator iii$ linear iii$
48  depend x,xyt$ depend y,xyt$ depend t,xyt$
49  let {iii(~aa*df(b(~bb),y),xyt)  =>
50      -df(aa,y)*b(bb)+fs*aa*b(~bb),
51      iii(~aa*df(b(~bb),x),xyt)  =>
52      -df(aa,x)*b(bb)-fs*df(vh,x)*aa*b(~bb),
53      iii(~aa*df(b(~bb),t),xyt)  =>
54      -df(aa,t)*b(bb)-fs*df(vh,t)*aa*b(~bb),
```

```
55      iii(~aa*df(b(~bb),y,2),xyt)=>
56      -iii(df(aa,y)*df(b(bb),y),xyt)+fs*aa*df(b(~bb),y),
57      iii(~aa*df(b(~bb),x,2),xyt)=>
58      -iii(df(aa,x)*df(b(bb),x),xyt)
59      -fs*df(vh,x)*aa*df(b(~bb),x),
60      iii(a(~aa)*b(~bb)*~cc,xyt)    => a(aa)*b(bb)*cc}$
61  iadj:=-(iii(a(u)*umom+a(v)*vmom+a(p)*cty,xyt)
62       +fs*a(h)*fkin)$
63  % extract adjoint PDEs
64  umom:=df(sub(fs=0,iadj),b(u))$
65  vmom:=df(sub(fs=0,iadj),b(v))$
66  cty:=df(sub(fs=0,iadj),b(p))$
67  % extract the adjoint FS boundary conditions
68  let {df(b(v),y)=>-df(b(u),x)}$
69  % df(u,y)=buy on the surface
70  depend f1,x,t$ depend f2,x,t$ depend f3,x,t$
71  depend f4,x,t$ depend f5,x,t$ depend f6,x,t$
72  depend f7,x,t$ depend f8,x,t$ depend f9,x,t$
73  buy:=-df(b(v),x)+f1*df(b(u),x)+f2*df(b(h),x)+f3*b(h)$
74  bp:=f4*df(b(u),x)+f5*(df(b(v),x)+buy)+f6*b(h)+f7*df(b(h),x)
75      +f8*df(b(h),x,2)$
76  iadj:=sub(b(p)=bp,sub(df(b(u),y)=buy,df(iadj,fs)))$
77  operator ii$ linear ii$
78  depend x,xt$ depend t,xt$
79  let {ii(~aa*df(b(~bb),x),xt)  =>-df(aa,x)*b(bb),
80      ii(~aa*df(b(~bb),y),xt)  => aa*df(b(bb),y),
81      ii(~aa*df(b(~bb),t),xt)  =>-df(aa,t)*b(bb),
82      ii(~aa*df(b(~bb),x,2),xt)  =>df(aa,x,2)*b(bb),
83      ii(~aa*b(~bb),xt)    => aa*b(bb)}$
84  iadj:=ii(iadj,xt)$
85  factor a$
86  b1:=df(iadj,b(h))$ b2:=df(iadj,b(u))$ b3:=df(iadj,b(v))$
87  % define coefficients entering the definition of buy
88  f1:=-coeffn(ftan,df(b(u),x),1)$
89  f2:=-coeffn(ftan,df(b(h),x),1)$
90  f3:=-coeffn(ftan,b(h),1)$
91  f4:=-coeffn(fnor,df(b(u),x),1)$
92  f5:=-coeffn(fnor,df(b(v),x),1)$
93  f6:=-coeffn(fnor,b(h),1)$
94  f7:=-coeffn(fnor,df(b(h),x),1)$
95  f8:=-coeffn(fnor,df(b(h),x,2),1)$
```

```
 96  % output for the resulting equations
 97  off nat; out "adj.red";
 98  umom; vmom; cty;
 99  b1; b2; b3;
100  write "end"; shut "adj.red"; on nat;
101  end$
102
103
```

Below we comment on the listed REDUCE program:

1. Preliminaries.

   - ℓ 10–12 describes states slightly displaced from the centre manifold.

   - ℓ 13–20 defines short hands and their properties for the expressions entering the free surface boundary conditions and appropriate algebraic and differential rules.

   - ℓ 22–33 expresses the physical fluid equations and their boundary conditions.

2. The linearisation about the centre manifold.

   - ℓ 35 makes use of the continuity equation to get rid of v_y and affirms that the free surface form η' does not depend on the vertical coordinate y.

   - ℓ 37, 38 state that $v'_y = -u'_x$ in order to simplify the boundary conditions. This definition has to be local to allow for the derivation of the adjoint continuity equation later. Thus ℓ 41 clears this rule after it is used here.

   - ℓ 36, 39 and 40 extracts the linearized kinematic and boundary conditions taking into account the variation in y of the free surface itself.

   - ℓ 44 extracts the linearized momentum and continuity equations.

3. Determination of the adjoint equations.

   - ℓ 47 introduces the operator iii which obtains the adjoints to the differential sub-operators entering linearized equations (16)–(20) through the integration by parts rules listed in ℓ 49–60. Note that the volumetric integrals in (22) after integration by parts contribute to the surface integral because the adjoint functions

and their derivatives generally are not zero on the free surface which is a function of x and t. In addition, note that the rule in ℓ 53–54 uses the rule previously defined in ℓ 42.

- ℓ 61–62 forms the inner product (22) taken with a negative sign for further convenience.

- Finally, ℓ 64–66 extracts the adjoint equations which result directly in the expression for the dual to be solved to obtain the model initial condition generating functions.

4. Determination of the adjoint kinematic and boundary conditions.

   This is done in three steps:

   - Firstly, the y derivatives of the unknown functions must be eliminated from the expressions for the adjoint boundary conditions since they remain undefined under the surface integration. This is done by making use of the continuity equation to eliminate v'_y (ℓ 68–76) and the tangential stress boundary condition (ℓ 73 with so far not defined coefficients f_i) to eliminate u'_y. Secondly, $p'(x, \eta + \eta', t)$ is eliminated through the normal stress boundary condition (ℓ 74–75).

   - The surface operator ii is introduced in ℓ 77, which specifies the integration by parts rules (ℓ 79–83) along the free surface. The adjoint boundary conditions are obtained by acting with the operator ii on the redefined in ℓ 76 inner product iadj (ℓ 84–86).

   - Finally, the undetermined coefficients f_i are determined in ℓ 88–95 and the final output is written in the separate file (ℓ 97–100).

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